Proofs from the Book:
"Several proofs of Pythagorean theorem"

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In mathematics, the Pythagorean theorem or Pythagoras' theorem, is a relation in Euclidean geometry between the three sides of a right triangle. The theorem is named after and commonly attributed to the 6th century BC Greek philosopher and mathematician Pythagoras, although the facts of the theorem were known by Indian (Baudhayana's and Katyayana's Sulbasutras), Greek, Chinese and Babylonian mathematicians well before he lived. Two contemporary proofs can be considered the oldest record of the Pythagorean theorem: one to be found in Chou Pei Suan Ching (The Arithmetical Classic of the Gnomon and the Circular Paths of Heaven, ca. 500-200 B.C.), the other in the Euclid's Elements.

Theorem 1 (Pythagorean theorem) *The sum of the areas of the squares on the legs of a right triangle is equal to the area of the square on the hypotenuse.*

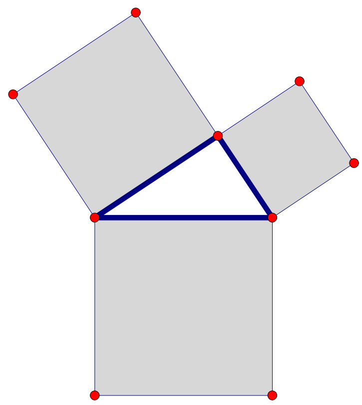

Figure 1: Pythagorean theorem

Using algebra, one can reformulate the theorem into its modern expression by noting that the area of a square is the square (second power) of the length of its side.

Theorem 2 (Pythagorean theorem) *Given a right triangle with legs of lengths a and b and hypotenuse of length c, then $a^2 + b^2 = c^2$.*

Proof 1. The square has a square hole with the side $(a-b)$. Summing up its area $(a-b)^2$ and $2ab$, the area of the four triangles $4 * \frac{ab}{2}$, we get

$$c^2 = (a-b)^2 + 2ab = a^2 - 2ab + b^2 + 2ab = a^2 + b^2.$$

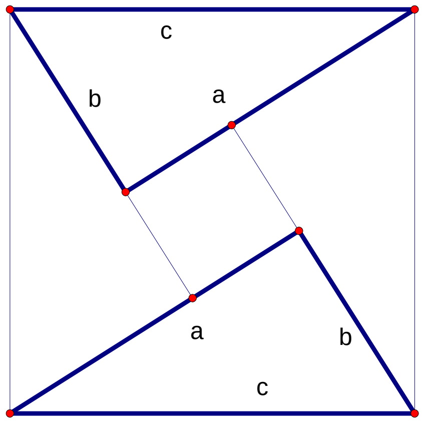

Figure 2: Proof 1.

Proof 2. We can compute the area of the big square in two ways. Thus $(a+b)^2 = \frac{4ab}{2} + c^2$ simplifying which we get the needed identity.

Proof 3. We start with the original triangle, now denoted ABC, and need only one additional construct - the altitude AD. The triangles ABC, BDA and ADC are similar which leads to two ratios:

$$\frac{AB}{BC} = \frac{BD}{AB}, \ \frac{AC}{BC} = \frac{DC}{AC}.$$

Written another way these become

$$AB * AB = BD * BC, \ AC * AC = DC * BC.$$

Summing up we get

$$AB * AB + AC * AC = BD * BC + DC * BC = (BD + DC) * BC = BC * BC.$$

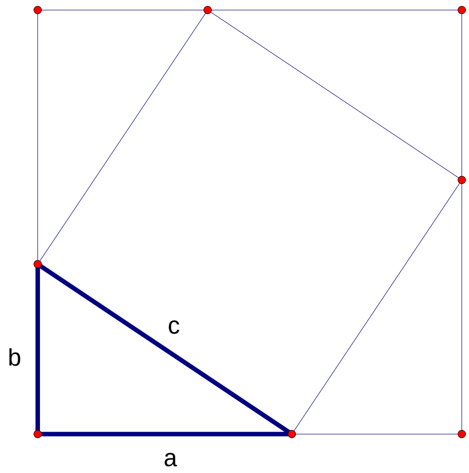

Figure 3: Proof 2.

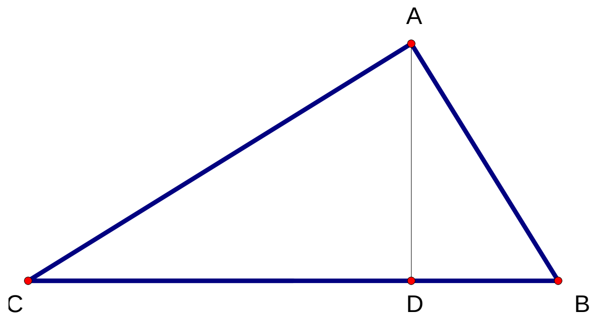

Figure 4: Proof 3.

Proof 4. Draw a circle with radius c and a right triangle with sides a and b as shown. In this situation, tree points F, G, H located on the circle form another right triangle with the altitude FK of the length a. It's hypotenuse GH is split in the ratio $\frac{c+b}{c-b}$. So, we get

$$a^2 = (c+b)(c-b) = c^2 - b^2.$$

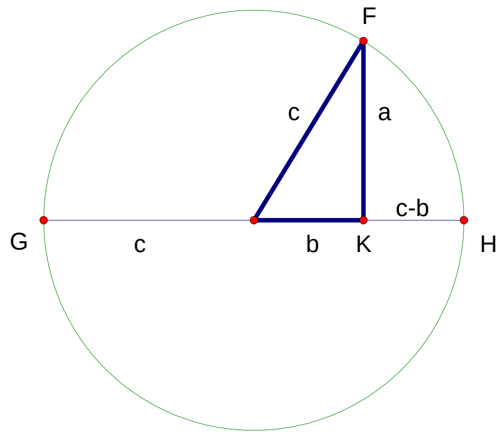

Figure 5: Proof 4.

Proof 5. On the small side AB add a right-angled triangle ABD similar to ABC. Then, naturaly, DBC is similar to other two. From area(ABD) +area(ABC)=area(DBC), $AD = \frac{AB^2}{AC}$ and $BD = \frac{AB*BC}{AC}$ we derive

$$\frac{AB^2}{AC} * AB + AB * AC = \frac{AB * BC}{AC} * BC.$$

Dividing by $\frac{AB}{AC}$ leads to $AB^2 + AC^2 = BC^2$.

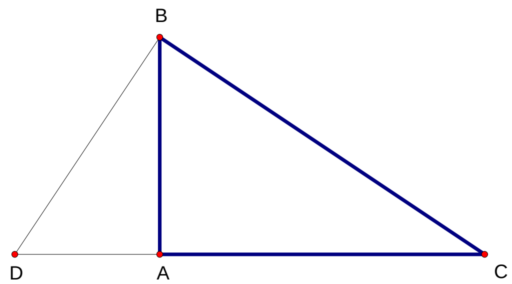

Figure 6: Proof 5.

Proof 6. Let ABC and DEF be two congruent right triangles such that B lies on DE and A, F, C, E are colinear. $BC = EF = a, AC = DF = b, AB = DE = c$. Obviously, $AB \perp DE$. Compute the area of $\triangle ADE$ in two diferent ways.

$Area(\triangle ADE) = \frac{AB*DE}{2} = \frac{c^2}{2}$ and also $Area(\triangle ADE) = \frac{DF*AE}{2} = \frac{b*AE}{2}$. $AE = AC + CE = b + CE$. CE can be found from similar triangles BCE and DFE: $CE = \frac{BC*FE}{DF} = \frac{a*a}{b}$. Putting things together we obtain

$$\frac{c^2}{2} = \frac{b(b + \frac{a^2}{b})}{2}.$$

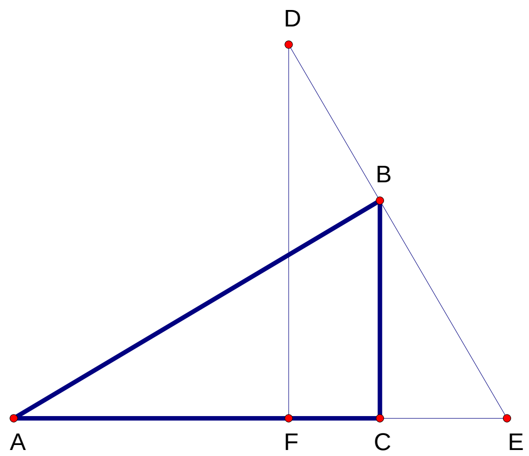

Figure 7: Proof 6.

Proof 7. Let $\angle ACB$ be right angle. As usual $AB = c$, $AC = b$ and $BC = a$. Define points D and E on AB so that $AD = AE = b$. By construcion, C lies on the circle with center A and radius b. Angle DCE subtends it's diametar and thus is right: $\angle DCE = 90$. It follows that $\angle BCD = \angle ACE$. Since $\triangle ACE$ is isosceles, $\angle CEA = \angle ACE$. Triangles DBC and EBC share $\angle DBC$. In addition, $\angle BCD = \angle BEC$. Therefore, triangles DBC and EBC are similar. We have $\frac{BC}{BE} = \frac{BD}{BC}$, or

$$\frac{a}{c+b} = \frac{c-b}{a}.$$

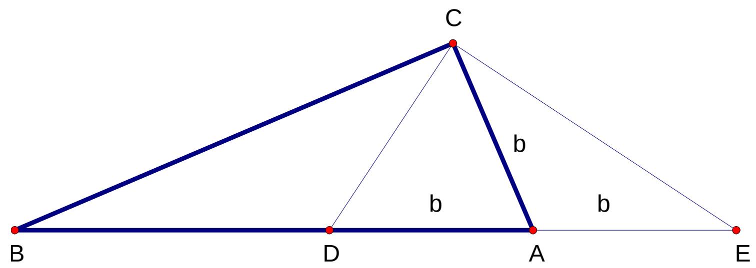

Figure 8: Proof 7.